Illustration - 21 *There are 6 single choice questions in an examination. How many sequence of answers are possible, if the first three questions have 4 choices each and the next three have 5 each ?*

(A) 15625 **(B)** 8000 **(C)** 4000 **(D)** 4096

SOLUTION : (B)

Here, we have to perform 6 jobs of answering 6 multiple choice questions. Each one of the first three questions can be answered in 4 ways and each one of the next three can be answered in 5 different ways.

So, the total number of different sequences $= 4 \times 4 \times 4 \times 5 \times 5 \times 5 = 8000$

COMBINATIONS — Section - 4

4.1 Definition of Combination

Let A, B, C be three different objects. The number of different ways in which we can select two objects out of A, B, C are AB, BC, CA.

These ways of selection of two objects from three different objects are also known as combinations of A, B, C taken two at a time or we can say grouping of A, B, C taken two at a time.

In permutation, we had seen that by changing the relative positions of the objects in the row we could generate new permutations *i.e.* permutation AB is different from BA. But when making combinations (groups or selections), by changing the relative positions of objects, we don't get new combinations.

For example : Combination (selection or group) of objects A, B is same as combination of objects B, A. Thus we treat AB and BA as same combination (selection or group).

4.2 Combination of *n* different objects taken *r* at a time without repetition of objects

We need to find number of ways to select (combine or group) r objects from n different objects.

Let n different objects be $a_1, a_2, a_3, \ldots\ldots, a_n$.

Let x be the number of ways to select r different objects. *i.e.* x represents number of different groups (combinations) of r objects that can be formed using n different objects.

Each group contains r different objects.

Now if we decide to permutate (arrange) r objects in a group, then it can be done in $\lfloor r$ ways.

Permutations of r different objects in x groups can be done in $x\lfloor r$ ways . . . (i)

But we know permutation of n different objects taken r at a time $= {}^nP_r$ . . . (ii)

Combining (i) and (ii), we get: $x\lfloor r = {}^nP_r$

$$\Rightarrow \quad x = \frac{{}^nP_r}{\lfloor r} = \frac{\lfloor n}{\lfloor n-r \lfloor r} = {}^nC_r \qquad \text{[Read it } {}^nC_r\text{]}$$

Number of ways to select r objects from n different objects $= {}^nC_r = \dfrac{\lfloor n}{\lfloor r \lfloor n-r}$.

Note : In new notation system, ${}^nC_r$ is also written as $C(n\,;\,r)$ or $\binom{n}{r}$

4.3 Properties of ^nC_r

(i) $^nC_0 = {^nC_n} = 1$

(ii) $^nC_r = {^nC_{n-r}}$

(iii) If $^nC_r = {^nC_k}$, then $r = k$ or $n - r = k$

(iv) $^nC_r + {^nC_{r-1}} = {^{n+1}C_r}$

(v) $r \cdot {^nC_r} = n\, {^{n-1}C_{r-1}}$

(vi) $\dfrac{1}{r+1}\,{^nC_r} = \dfrac{1}{n+1}\,{^{n+1}C_{r+1}}$

(vii) $\dfrac{^nC_r}{^nC_{r-1}} = \dfrac{n-r+1}{r}$

(viii) (a) If n is even, ^nC_r is greatest for $r = n/2$.

(b) If n is odd, ^nC_r is greatest for $r = \dfrac{n-1}{2}, \dfrac{n+1}{2}$.

Illustration - 22 *Select the correct choice from the given choices of following questions.*

(i) *How many triangles can be formed by joining the vertices of a hexagon ?*

(A) 10 (B) 20 (C) 30 (D) 60

(ii) *How many diagonals are there in a polygon with n sides ?*

(A) $\dfrac{n(n-1)}{2}$ (B) $\dfrac{n(n+1)}{2}$ (C) $\dfrac{n(n-3)}{2}$ (D) $\dfrac{n(n+3)}{2}$

SOLUTION : (i).(B) (ii).(C)

(i) Let $A_1, A_2, A_3, \ldots\ldots, A_6$ be the vertices of the hexagon. One triangle is formed by selecting a group of 3 points from 6 given vertices.

Number of triangles = Number of groups of 3 each from 6 points.

$$= {^6C_3} = \frac{6!}{3!\ 3!} = 20$$

(ii) Number of lines that can be formed by using the given vertices of a polygon

= Number of groups of 2 points each selected from the n points.

$$= {^nC_2} = \frac{n!}{2!\ (n-2)!} = \frac{n(n-1)}{2}$$

➢ Out of $^nC_2$ lines, n are the sides of the polygon and remaining $^nC_2 - n$ are the diagonals.

So, number of diagonals $= \dfrac{n(n-1)}{2} - n = \dfrac{n(n-3)}{2}$

Illustration - 23 *In how many ways can a cricket team be selected from a group of 25 players containing 10 batsmen, 8 bowlers, 5 all-rounders and 2 wicketkeepers ? Assume that the team of 11 players requires 5 batsmen, 3 all-rounders, 2-bowlers and 1 wicketkeeper.*

(A) $\dfrac{10!}{5!}$ (B) $\dfrac{14 \times 10!}{5!}$ (C) $\dfrac{14 \times 10!}{3 \times 5!}$ (D) $\dfrac{10!}{3 \times 5!}$

SOLUTION : (C)

Divide the selection of team into four operations.

I : Selection of batsmen can be done (5 from 10) in $^{10}C_5$ ways.

II : Selection of bowlers can be done (2 from 8) in $^{8}C_2$ ways.

III : Selection of all-rounders can be done (3 from 5) in $^{5}C_3$ ways.

IV : Selection of wicketkeeper can be done (1 from 2) in $^{2}C_1$ ways.

Hence, the team can be selected in $= {}^{10}C_5 \times {}^{8}C_2 \times {}^{5}C_3 \times {}^{2}C_1$ ways $= \dfrac{14 \times 10!}{3 \times 5!}$ ways.

Illustration - 24 *A man has 7 relatives, 4 of them ladies and 3 gentlemen ; his wife has 7 relatives, 3 of them are ladies and 4 gentlemen. In how many ways can he invite a dinner party of 3 ladies and 3 gentlemen so that there are 3 of man's relatives and 3 of wife's relatives?*

(A) 144 **(B)** 720 **(C)** 485 **(D)** 340

SOLUTION : (C)

The possible ways of selecting 3 ladies and 3 gentlemen for the party can be analysed with the help of the following table.

Man's relative		Wife's relative		Number of ways
Ladies (4)	Gentlemen (3)	Ladies (3)	Gentlemen (4)	
3	0	0	3	${}^4C_3\, {}^3C_0\, {}^3C_0\, {}^4C_3 = 16$
2	1	1	2	${}^4C_2\, {}^3C_1\, {}^3C_1\, {}^4C_2 = 324$
1	2	2	1	${}^4C_1\, {}^3C_2\, {}^3C_2\, {}^4C_1 = 144$
0	3	3	0	${}^4C_0\, {}^3C_3\, {}^3C_3\, {}^4C_0 = 1$

Total number of ways to invite = 16 + 324 + 144 + 1 = 485

Illustration - 25 *A tea party is arranged for 16 people along two sides of a long table with 8 chairs on each side. Four men wish to sit on one particular side and two on the other side. In how many ways can they be seated ?*

(A) $\dfrac{8!\,10!}{4!\,6!}$ **(B)** $\dfrac{8!\,8!\,10!}{4!\,6!}$ **(C)** $\dfrac{8!\,8!}{4!\,6!}$ **(D)** $\dfrac{8!\,8!}{6!}$

SOLUTION : (B)

Let $A_1, A_2, A_3, \ldots\ldots, A_{16}$ be the sixteen persons. Assume that A_1, A_2, A_3, A_4 want to sit on side 1 and A_5, A_6 want to sit on side 2.

The persons can be made to sit if we complete the following operations.

(i) Select 4 chairs from the side 1 in $^8C_4$ ways and allot these chairs to A_1, A_2, A_3, A_4 in 4 ! ways.

(ii) Select two chairs from side 2 in $^8C_2$ ways and allot these two chairs to A_5, A_6 in 2 ! ways.

(iii) Arrange the remaining 10 persons in remaining 10 chairs in 10 ! ways.

$\Rightarrow$ Hence the total number of ways in which the persons can be arranged

$$= \left({}^8C_4\, 4!\right)\left({}^8C_2\, 2!\right)\left(10!\right) = \frac{8!}{4!\,4!}\,4! \times \frac{8!\,2!}{2!\,6!}\,10! = \frac{8!\,8!\,10!}{4!\,6!}$$

Illustration - 26 *A mixed doubles tennis game is to be arranged from 5 married couples. In how many ways the game be arranged if no husband and wife pair is included in the same game?*

(A) 60 **(B)** 30 **(C)** 120 **(D)** 240

SOLUTION : (A)

To arrange the game we have to do the following operations.

(i) Select two men from 5 men in ${}^5C_2$ ways.

(ii) Select two women from 3 women excluding the wives of the men already selected. This can be done in ${}^3C_2$ ways.

(iii) Arrange the 4 selected persons in two teams. If the selected men are M_1 and M_2 and the selected women are W_1 and W_2 , this can be done in 2 ways :

M_1W_1 play against M_2W_2

M_2W_1 play against M_1W_2

Hence, the number of ways to arrange the game $= {}^5C_2\ {}^3C_2\ (2) = 10 \times 3 \times 2 = 60$

TYPICAL PROBLEM CATEGORIES SECTION - 5

Most of the typical problems in permutation and combination can be categorised in various types. To solve these typical problems, it is advisable to apply standard ways or methods available.

Let us call these categories of problems as Typical Problem Categories (TPC).

5.1 TPC - 1 : Always including particular objects in the selection

The number of ways to select r objects from n different objects where p particular objects should always be included in the selection $= {}^{n-p}C_{r-p}$.

Logic :

We can select p particular objects in 1 way. Now from remaining $(n - p)$ objects we select remaining $(r - p)$ objects in ${}^{n-p}C_{r-p}$ ways.

Using fundamental principle of counting, number of ways to select r objects where p particular objects are always included $= 1 \times {}^{n-p}C_{r-p} = {}^{n-p}C_{r-p}$.

***Illustrating the Concepts* :**

In how many ways a team of 11 players be selected from a list of 16 players where two particular players should always be included in the team.

Using formula given in TPC-1, number of ways to make a team of 11 players from 16 players always including 2 particular players $= {}^{16-2}C_{11-2} = {}^{14}C_9$